A point based generalization of the spatial logic RCC

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Abstract

We present a point based qualitative spatial logic for representing and reasoning about spatial objects and regions. The logic can simultaneously represent objects of different dimensions and is a generalization of RCC.

Main Topic: Knowledge representation.

Keywords: Qualitative first order spatial logic, RCC.

1 Introduction

Randell, Cui, and Cohn's [1, 3] qualitative spatial logic is popular and influential in AI. This logic is commonly referred to as RCC. RCC is a qualitative logic for representing spatial objects and regions. One limitation of this logic is that it can only represent N-Dimensional objects in N-Dimensional space.

We present a new spatial logic based on points. Our logic allows the user to represent RCC's spatial relations without restricting the dimensionality of the objects and the space.

2 Point-set topology

We adopt the standard point-set topological definitions presented in [2, p.164–165]:

> This model of topological spatial relations is based on the point-set topological notions of *interior* and *boundary.* In this section the appropriate definitions and results from point-set topology are presented.
>
> Let X be a set. A topology on X is a collection A of subsets of X that satisfies the three conditions: (1) the empty set and X are in A; (2) A is closed under arbitrary unions; and (3) A is closed under finite intersections. A topological space is a set X with a topology A on X. The sets in a topology on X are called open sets, and their complements in X are called closed sets. The collection of closed sets: (1) contains the empty set and X; (2) is closed under arbitrary intersections; and (3) is closed under finite unions.
>
> Via the open sets in a topology on a set X, a set-theoretic notion of closeness is established. If U is an open set and $x \in U$, then U is said to be a *neighborhood* of x.
>
> Given $Y \subset X$, the *interior* of Y, denoted by Y^o, is defined to be the union of all

open sets that are contained in Y, i.e., the interior of Y is the largest open set contained in Y. y is in the interior of Y if and only if there is a neighborhood of y contained in Y, i.e., $y \in Y^o$ if, and only if, there is an open set U such that $y \in U \subset Y$. The interior of a set could be empty, e.g., the interior of the empty set is empty. The interior of X is X itself. If U is open then $U^o = U$. If $Z \subset Y$ then $Z^o \subset Y^o$.

The *closure* of Y, denoted by $\overline{Y}$, is defined to be the intersection of all closed sets that contain Y, i.e., the closure of Y is the smallest closed set containing Y. It follows that y is in the closure of Y if and only if every neighborhood of y intersects Y, i.e., $y \in \overline{Y}$ if and only if $U \cap Y \neq \emptyset$ for every open set U containing y. The empty set is the only set with empty closure. The closure of X is X itself. If C is closed then $\overline{C} = C$. If $Z \subset Y$ then $\overline{Z} \subset \overline{Y}$.

The *boundary* of Y, denoted by δY, is the intersection of the closure of Y and the closure of the complement of Y, i.e., $\delta Y = \overline{Y} \cap \overline{X - Y}$. The boundary is a closed set. It follows that y is in the boundary of Y if and only if every neighborhood of y intersects both Y and its complement, i.e., $y \in \delta Y$ if and only if $U \cap Y \neq \emptyset$ and $U \cap (X - Y) \neq \emptyset$ for every open set U containing y. The boundary can be empty, e.g., the boundaries of both X and the empty set are empty.

The relationships between interior, closure and boundary are described by the following propositions:

$$Y^o \cap \delta Y = \emptyset,$$

$$Y^o \cup \delta Y = \overline{Y}.$$

Note that the interior and boundary of an object depends on its topological space X. For example, the boundary of a line in 1-dimensional space is its endpoints with the rest of the line being its interior. In 2-dimensional space, the boundary is the whole line, and the interior is empty.

3 Axiomatization

3.1 Boundary and interior

We define a point based sorted first order spatial logic. The logic is built by specifying which points lie on the boundary and interior of an object.

The relation $boundary(p, r)$ is true if and only if point p is on the boundary of region r (i.e., $p \in \delta r$). Similarly, the relation $interior(p, r)$ is true if and only if point p is in the interior of region r (i.e., $p \in r^o$). The relation $closure(p, r)$ is true if and only if point p is in the closure of region r (i.e., $p \in \overline{r}$). Only a point can appear as the first parameter and may appear as the second parameter.

3.2 Open and closed regions

When defining a temporal logic, it is common practice to not make any commitments at the endpoints of an interval (e.g., see [5]). For example, if we want to specify that the house is red between times t_1 and t_2 we could write $true(t_1, t_2, house(red))$. This is intended to specify that the house is red at all times between t_1 and t_2, and no color commitments are made at times t_1 and t_2. In other words, the interval from t_1 to t_2 may be open, closed, or open at one end and closed at the other. Note that we use t_1 and t_2 to specify the interval even though they may not be part of the interval.

We define spatial intervals analogously. Every spatial entity has a boundary, even though this boundary may be empty or not part of the object (i.e., the object may be open). To distinguish between open and closed objects, we introduce the relation $partOf(p, r)$ which is true if and only if point p is a part of object r (i.e., $p \in r$) . We have the axiom:

$$\forall p, r.\ \ interior(p, r) \rightarrow partOf(p, r)$$

that specifies that the interior of an object is always part of an object. If object o is open:

$$\forall p.\ \ boundary(p, o) \rightarrow \neg partOf(p, o),$$

and if it is closed:

$$\forall p.\ \ boundary(p, o) \rightarrow partOf(p, o).$$

3.3 The empty and universal regions

We introduce two constants EMPTY and UNIVERSAL (i.e., X) to represent the empty and universal regions respectively. The EMPTY region has no boundary or interior:

$$\forall p.\ \ \neg boundary(p, EMPTY) \quad \wedge \quad \neg interior(p, EMPTY).$$

The UNIVERSAL region has everything in its interior and no boundary:

$$\forall p.\ \ \neg boundary(p, UNIVERSAL) \quad \wedge \quad interior(p, UNIVERSAL).$$

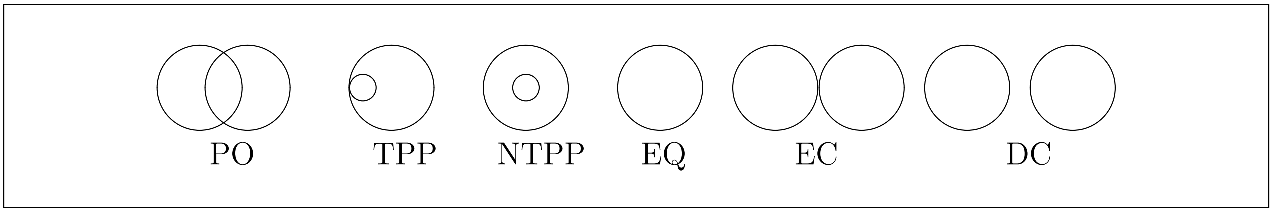


Figure 1: Possible spatial relations

3.4 Spatial relations

In [1, 3], Randell, Cui, and Cohn present a set of 8 diadic spatial relationships that are mutually exhaustive and pair-wise disjoint. The relations are shown in figure 1. TPP and NTPP have inverses labelled TPP^{-1} and $NTPP^{-1}$.

Our axiomatization of the 8 spatial relations in terms of *boundary* and *interior* is given in figure 2. In order to maintain the property that there is only one relationship between two regions, we added the following to the axiomatization:

- The third conjunct in axiom (2) (i.e., $\neg EQ(x, y)$) is required to make $EC(p, p)$ false when p is a point (we have $EQ(p, p)$ for a point p). Similarly, $\neg NTPP(x, y)$ is required to make $EC(x, y)$ false in situations such as we have x is a line and y is a plane in 2-D space, and x lies within y. In this case, we have $NTPP(x, y)$.

- The second line in axiom (1) is required to prevent the EMPTY region from participating in the DC relationship since the first conjunct is always true when x or y is EMPTY. As shown below, EMPTY is NTPP with every other region.

Note that whether a region is open or closed is irrelevant to our axiomatization.

We have the theorems that every region is a non-tangential proper part of UNIVERSAL:

$\forall x.\ NTPP(x, UNIVERSAL)$,

$$\forall x,y.\;\; EQ(x,y) \leftrightarrow [\forall p.\;\; partOf(p,x) \leftrightarrow partOf(p,y)].$$

$$\begin{aligned} \forall x,y.\;\; DC(x,y) \;\;\leftrightarrow\;\; & [(\neg\exists p.\;\; closure(p,x) \wedge closure(p,y)) \\ & \wedge\;\; \neg EQ(x,EMPTY) \;\;\wedge\;\; \neg EQ(y,EMPTY)]. \end{aligned} \tag{1}$$

$$\forall x,y.\;\; NTPP(x,y) \leftrightarrow [\forall p.\;\; closure(p,x) \rightarrow interior(p,y)].$$

$$\forall x,y.\;\; NTPP^{-1}(x,y) \leftrightarrow NTPP(y,x).$$

$$\begin{aligned} \forall x,y.\;\; EC(x,y) \;\;\leftrightarrow\;\; & [(\exists p_1.\;\; closure(p_1,x) \wedge closure(p_1,y)) \\ & \wedge\;\; (\neg\exists p_2.\;\; interior(p_2,x) \wedge interior(p_2,y)) \\ & \wedge\;\; \neg EQ(x,y) \;\;\wedge\;\; \neg NTPP(x,y)]. \end{aligned} \tag{2}$$

$$\begin{aligned} \forall x,y.\;\; PO(x,y) \;\;\leftrightarrow\;\; & [(\exists p_1.\;\; interior(p_1,x) \wedge interior(p_1,y)) \\ & \wedge\;\; (\exists p_2.\;\; interior(p_2,x) \wedge \neg interior(p_2,y)) \\ & \wedge\;\; (\exists p_3.\;\; \neg interior(p_3,x) \wedge interior(p_3,y))]. \end{aligned}$$

$$\begin{aligned} \forall x,y.\;\; TPP(x,y) \;\;\leftrightarrow\;\; & [(\forall p_1.\;\; closure(p_1,x) \rightarrow closure(p_1,y)) \\ & \wedge\;\; (\exists p_2.\;\; boundary(p_2,x) \wedge interior(p_2,y)) \\ & \wedge\;\; (\exists p_3.\;\; boundary(p_3,x) \wedge boundary(p_3,y))]. \end{aligned}$$

$$\forall x,y.\;\; TPP^{-1}(x,y) \leftrightarrow TPP(y,x).$$

Figure 2: Point based axiomatization of the RCC relations

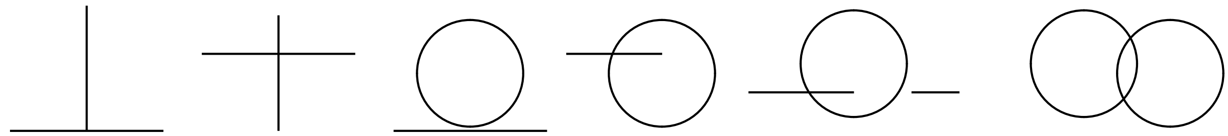


Figure 3: EC examples

and EMPTY is a non-tangential proper part of every region (c.f., the empty set is a subset of every set):

$$\forall x.\ \ NTPP(EMPTY, x). \tag{3}$$

A lemma of theorem (3) is:

$$\forall x \exists y.\ \ NTPP(y, x)$$

which is also a theorem in [3].

Our axiomatization in figure 2 places no restrictions on the dimensions of the objects involved. For example, the axioms can be used to describe the spatial relationships between a point and a cube. The allowable relationships across dimensions are shown in table 1. Each row $x - y$ in table 1 lists the possible spatial relations between an object of dimension x and y in y-D space. For example, the row 0-1 gives the 3 possible spatial relationships between a point and a line. An empty entry indicates that the relation cannot occur between the objects involved. Note that the line segments in row 1-1 are supposed to appear on a single line. Also note that the only rows that are completely filled in are the ones that involve objects of equal non-zero dimensions.

Interesting EC relationships are shown in figure 3. Diagram A has the end of one line

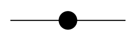
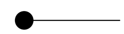
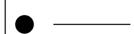
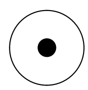
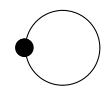
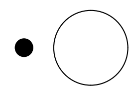
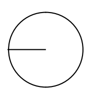
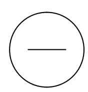
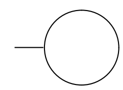
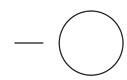
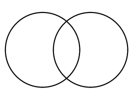
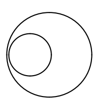
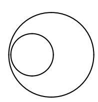
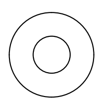
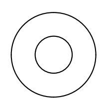
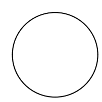
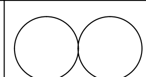
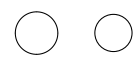
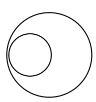
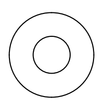
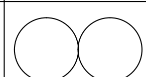
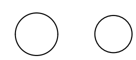
Dim	PO	TPP	TPP $^{-1}$	NTPP	NTPP $^{-1}$	EQ	EC	DC
0-1								
0-2 0-3								
1-1								
1-2 1-3								
2-2 3-3								
2-3								

Table 1: Spatial relations across dimensions

intersecting with the middle of another. In diagram B, we have two lines intersecting in their interiors. Diagram C has a line touching a disk (A,B and C are in 2-D space). If the line overlaps the same 2-D disk in 3-D space, they are EC (diagram D). Diagram E represents an arrow shot through a disk (the objects are perpendicular to each other) in 3-D space. Diagram F has a 3-D sphere on the left intersecting with a 2-D disk on the right.

3.5 Functions

The basic binary functions between regions for union, intersection, complement, and difference are axiomatized in figure 4. Theorems which follow from the axiomatization are given in figure 5.

Axiom (4) specifies than an object and its complement share the same boundary. It directly follows from this that they are connected (i.e., theorem (5)). Note that theorem (5) is also a theorem in [3].

4 Topological transition graphs

Over time, it is possible for the spatial relation between two objects to change (i.e., a topological transition occurs). For example, two billiard balls which approach each other and then collide go from a DC (disconnected) to an EC (touching) relationship.

There are restrictions on the topological transitions. It is not always the case that the spatial relation between two objects can change to any one of the other relations. One factor which restricts the topological transitions is the dimension of the objects involved. Recall that in table 1, not all spatial relations are possible between objects of different

$$\forall p, x, y.\ \ interior(p, union(x, y)) \ \leftrightarrow\ [interior(p, x) \vee interior(p, y)].$$

$$\forall p, x, y.\ \ boundary(p, union(x, y)) \ \leftrightarrow\ [boundary(p, x) \vee boundary(p, y)] \\ \wedge \neg interior(p, x) \wedge \neg interior(p, y).$$

$$\forall p, x, y.\ \ interior(p, intersection(x, y)) \ \leftrightarrow\ [interior(p, x) \wedge interior(p, y)].$$

$$\forall p, x, y.\ \ boundary(p, intersection(x, y)) \ \leftrightarrow\ [boundary(p, x) \wedge boundary(p, y)] \\ \vee\ [boundary(p, x) \wedge interior(p, y)] \\ \vee\ [boundary(p, y) \wedge interior(p, x)].$$

$$\forall p, x.\ \ interior(p, complement(x)) \ \leftrightarrow\ \neg closure(p, x).$$

$$\forall p, x.\ \ boundary(p, complement(x)) \ \leftrightarrow\ boundary(p, x). \tag{4}$$

$$\forall p, x, y.\ \ interior(p, difference(x, y)) \ \leftrightarrow\ [interior(p, x) \wedge \neg closure(p, y)].$$

Figure 4: Binary function axiomatization

$$
\begin{gathered}
\forall x.\;\; EQ(union(UNIVERSAL, x), UNIVERSAL). \\
\forall x.\;\; EQ(intersection(UNIVERSAL, x), x). \\
\forall x.\;\; EQ(difference(UNIVERSAL, x), complement(x)). \\
\forall x.\;\; EQ(difference(x, UNIVERSAL), EMPTY). \\
EQ(complement(UNIVERSAL), EMPTY). \\
\forall x.\;\; EQ(union(EMPTY, x), x). \\
\forall x.\;\; EQ(intersection(EMPTY, x), EMPTY). \\
\forall x.\;\; EQ(difference(EMPTY, x), EMPTY). \\
\forall x.\;\; EQ(difference(x, EMPTY), x). \\
EQ(complement(EMPTY), UNIVERSAL). \\
\forall p, x, y.\;\; [partOf(p, x) \vee partOf(p, y)] \rightarrow partOf(p, union(x, y)). \\
\forall p, x, y.\;\; [partOf(p, x) \wedge partOf(p, y)] \rightarrow partOf(p, intersection(x, y)). \\
\forall x.\;\; EQ(union(x, x), x). \\
\forall x.\;\; EQ(intersection(x, x), x). \\
\forall x.\;\; EQ(complement(complement(x)), x). \\
\forall x.\;\; EC(complement(x), x). \qquad (5) \\
\forall p, x.\;\; \neg interior(p, difference(x, x)).
\end{gathered}
$$

Figure 5: Various theorems

dimensions. Another factor is the topological spatial dimension. For example assume we have two disconnected (i.e., DC) line segments of equal length. In one dimensional space, from DC the lines can only go to EC. But in two dimensional space, other transitions from DC are allowed. For example, we can position the lines so that they are parallel (i.e., still DC), then push them together. The spatial relation goes from DC to EQ. Such a transition is not possible in one dimensional space.

To represent the transitions, we use a *topological transition graph.* Each vertex is labelled with a spatial relation. An edge represents a topological transition between its incident vertices. The transition can occur in either direction.

Topological transition graphs change with the dimension of the objects and the space. The different graphs are listed in table 2. The left hand column gives the dimension of the objects involved. For example, entry "0-3" specifies that this row lists spatial graphs that apply only to objects of 0 and 3 dimensions. The top row lists the possible spatial dimensions. Letters in the table refer to the graphs in figure 6. If no topological transition graph exists, the entry in table 2 is blank.

Table 2 is used as follows. Suppose we want the topological transition graph between a line and a sphere. These are 1 and 3 dimensional objects respectively. We therefore go to the row labelled 1-3. A sphere can only exist in 3 dimensional space. We therefore go to the column labelled 3-D. The entry we find is B. The applicable graph is thus graph B in figure 6.

Note that graph F occurs in table 2 whenever we have N dimensional objects in N dimensional space. The logic RCC deals specifically with N dimensional objects in N dimensional space. Graph F coincides with the topological transition graph in [1, p. 172].

	1-D	2-D	3-D
0-0	A	A	A
0-1	B	D	D
0-2		B	D
0-3			B
1-1	F	E	E
2-2		F	E
3-3			F
1-2		B	D
1-3			B
2-3			B

Table 2: Topological transition graphs across dimensions

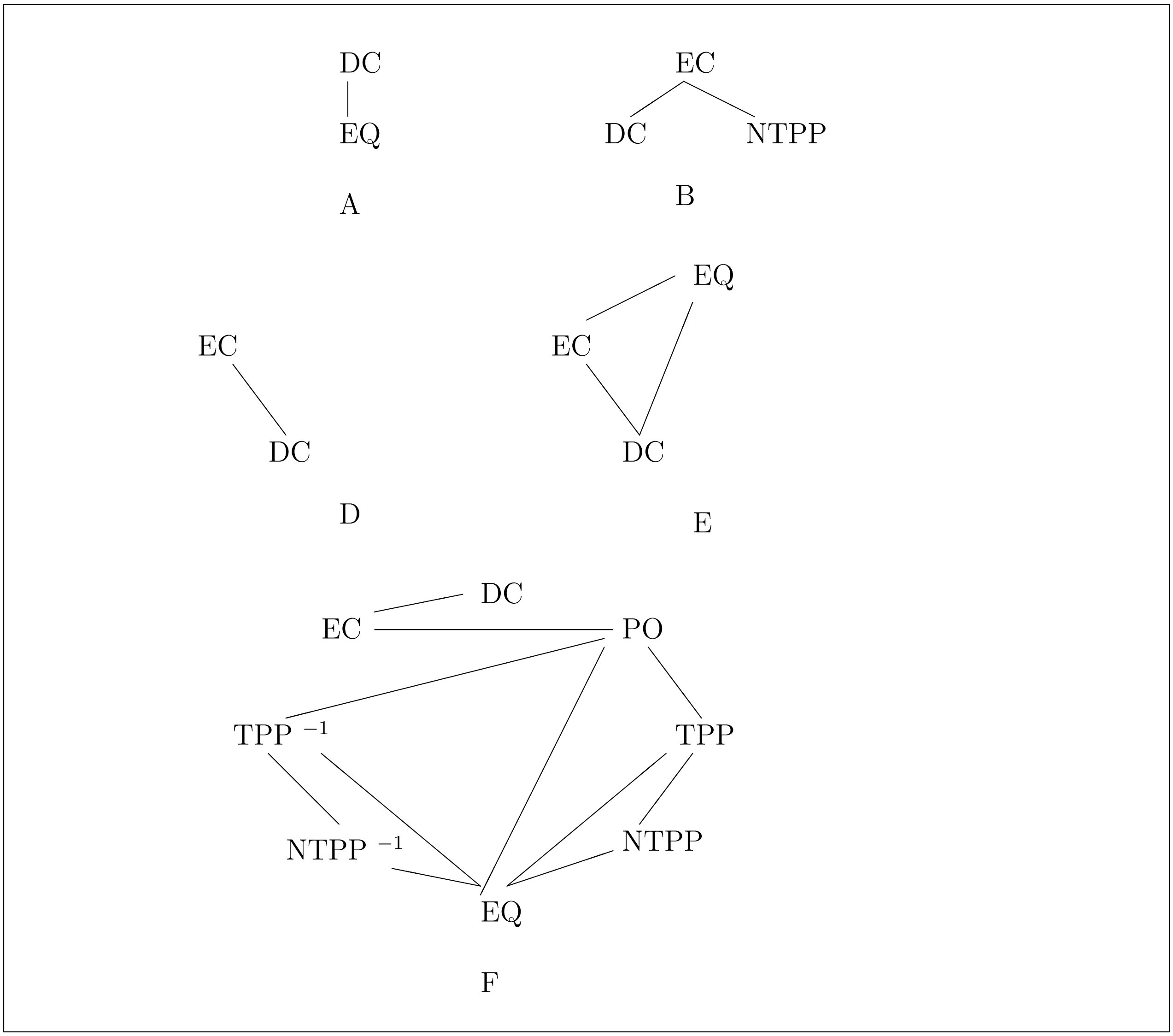


Figure 6: The various topological transition graphs

RCC relation	Topological relation
DC(a,b)	$\overline{a} \cap \overline{b} = \emptyset$
EC(a,b)	$(\overline{a})^o \cap (\overline{b})^o = \emptyset, \overline{a} \cap \overline{b} \neq \emptyset$
PO(a,b)	$(\overline{a})^o \cap (\overline{b})^o \neq \emptyset, \overline{a} \not\subseteq \overline{b}, \overline{b} \not\subseteq \overline{a}$
TPP(a,b)	$\overline{a} \subset \overline{b}, \overline{a} \not\subseteq (\overline{b})^o$
TPP^{-1}(a,b)	$\overline{b} \subset \overline{a}, \overline{b} \not\subseteq (\overline{a})^o$
NTPP(a,b)	$\overline{a} \subset (\overline{b})^o$
$NTPP^{-1}$(a,b)	$\overline{b} \subset (\overline{a})^o$
EQ(a,b)	$\overline{a} = \overline{b}$

Table 3: RCC's topological interpretation

5 Comparison with previous work

In [4, p. 71], Renz and Nebel present a point set topological interpretation of RCC which is shown in table 3. Their topological relations, as with RCC, are restricted to N-dimensional objects in N-dimensional space. Also, unlike RCC and our axiomatization, their topological relations are not mutually disjoint. For example, DC(a,b) and NTPP(a,b) both hold simultaneously when "a" is the empty set. Similarly, EC(a,b) and EQ(a,b) both hold when "a" and "b" both equal to the same point (i.e., the relations are not valid in 1-dimensional space).

6 Conclusion

We presented a point based sorted first order spatial logic capable of representing RCC's 8 spatial relations. The advantage of our approach is that objects and space are not restricted to the same dimension. For example, we can represent the spatial relations between a line and a plane in 3-D space. We can also represent the topological transition of spatial relations between objects of different dimensions across spatial dimensions.

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